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# Dynamics and thermodynamics of a gas of automata

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**Abstract** – We consider a system of point charges interacting within a cone of vision and confined by an external potential, as a simple model of individuals provided with vision. The non Newtonian nature of the interaction introduces dissipative effects which are balanced by a memory mechanism. The two-body system is amenable to quadrature, whereas the $N \gg 1$ body system exhibits crystal-like and disordered states with a non-trivial phase diagram if the interaction range and memory persistence are chosen as control parameters.



**Introduction.** – The interactions based on perception, as vision, do not obey the third principle of Newtonian dynamics. In this paper we study the effect that this non Newtonian behaviour has on the dynamics of a two-body and a many-body system.

We propose the gas of Von Neumann automata as a basic model for complex systems formed by a large number of interacting individuals provided with a sensory system, such as a crowd or a swarm [1]. With "gas of automata" we mean a system similar to a gas of charged particles in a continuous 2D space confined by an external potential, except for the presence of a perception system, described by a visual cone of range $r_v$ and angular half-aperture $\alpha$. An automaton $A$ is repelled by an automaton $B$ when it falls within its visual cone, and feels no interaction when $B$ is out of the cone (see fig. 1). Repulsion decreases with distance and we assume an inverse proportionality as in the 2D electrostatic case. More explicitly the force is given by

$$\mathbf{F} = \begin{cases} \dfrac{\mathbf{r}}{r^2}, & \text{if } B \in C_A, \\ 0, & \text{if } B \notin C_A, \end{cases} \qquad (1)$$

where $\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$ is the displacement and $r = |\mathbf{r}|$ is the distance from $A$ to $B$. The cone condition is defined by

$$B \in C_A \ \text{ if } \ r < r_v \ \text{ and if } \ |\theta| < \alpha \quad \cos\theta = -\frac{\mathbf{r}\cdot\mathbf{v}}{r\,v}, \qquad (2)$$

where $\mathbf{v}$ is the velocity of $A$, along which we choose the axis of the cone (fig. 1).

The model is supposed to describe a "level-zero approximation" of a low-density crowd in which the

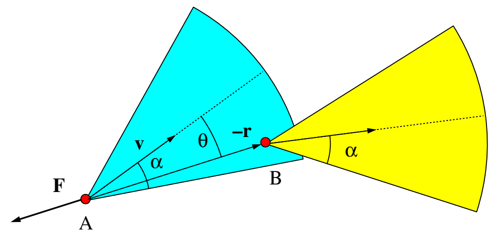


Fig. 1: Automaton $B$ falls in the cone of vision of automaton $A$, which feels repulsive force $\mathbf{F}$, while $B$ does not feel any force due to the presence of $A$.

automata ("pedestrians") tend to avoid each other, even though in an actual crowd the interactions are certainly more involved: social attractive forces, responsible of the formation of small clusters, and more complex behavioural patterns are certainly present in addition to pure misanthropy. The presence of the visual cone renders the force non-Newtonian and changes significantly the $N$ automata problem with respect to the $N$-body problem, by introducing a sort of damping (maximised in the $\alpha = \frac{\pi}{2}$ case), since the repulsive automaton-to-automaton force is mainly opposed to the direction of the motion and thus has negative power.

The model, given the simplicity of both the perception and decision systems of automata, and also the point-like nature of our automata, is not to be intended to be able of describing actual crowd dynamics, but is just a toy model that allows to investigate, using both analytical and numerical methods, the non-Newtonian features introduced by perception.

More realistic models able to describe traffic flow and crowd dynamics have been proposed, using descriptions based on cellular automata [2] and social forces [3], and also evolutionary methods based on genetic algorithms and neural networks [4].

**The two-automata problem.** – The two-automata problem can be solved by passing to the description of the relative motion which resembles the dynamics of a particle moving in a central time-dependent force field (the mathematical details can be found in [4]). Let us assume that the automata are subject to an external force field, for example a harmonic potential. We can write the forces felt by the automata as

$$\mathbf{F}_{12} = -\omega^2\,\mathbf{r}_1 + \frac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}^2}\ \vartheta(C_1), \tag{3}$$

$$\mathbf{F}_{21} = -\omega^2\,\mathbf{r}_2 + \frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{12}^2}\ \vartheta(C_2), \tag{4}$$

where the cone conditions are expressed (assuming for simplicity's sake $r_v = \infty$) by

$$C_1 = \mathbf{v_1}\cdot(\mathbf{r_2} - \mathbf{r_1}) - v_1\, r_{12}\ \cos\alpha, \tag{5}$$

$$C_2 = \mathbf{v_2}\cdot(\mathbf{r_1} - \mathbf{r_2}) - v_2\, r_{21}\ \cos\alpha, \tag{6}$$

and $\vartheta(u)$ is the step function

$$\vartheta(u) = \begin{cases} 0, & \text{if } u \leqslant 0,\\ 1, & \text{if } u > 0. \end{cases} \tag{7}$$

We can then pass to the centre-of-mass ($\mathbf{R}$,$\mathbf{V}$) and relative-distance ($\mathbf{r}$,$\mathbf{v}$) coordinates, and we obtain three different Hamiltonian functions for the relative motion, depending on the relative positions and velocities of the automata.
The Hamiltonian for the relative motion reads

$$H_I = H \equiv \frac{v^2}{2} + \omega^2\frac{r^2}{2} \tag{8}$$

when none of the automata sees the other one;

$$H_{II} = H + V(r) \quad V(r) \equiv -\ln(r) \tag{9}$$

when just one of them sees the other one;

$$H_{III} = H + 2V(r) \tag{10}$$

when they see each other. The time dependence of the potential is not explicit, but depends on the time evolution of the phase space trajectories of the automata. The energy is conserved in each zone $I$, $II$ and $III$, but not conserved in the sharp transitions between one zone and the other. Nevertheless, since the angular momentum for the relative motion $L \equiv \mathbf{r}\times\mathbf{v}$ is conserved, we can limit ourselves to the study of the radial motion, introducing an effective potential that changes with time, according to the cone condition, and the equation of motion can in principle be integrated, even if the integration is quite cumbersome due to the cone condition (the equations for the centre-of-mass motion *are not* separated to those for the relative motion, in contrast with the usual case).

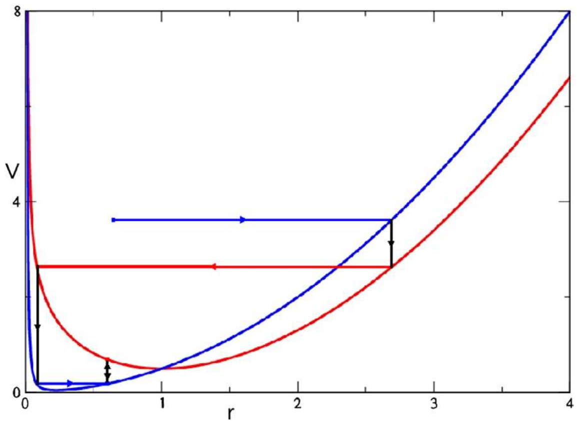


Fig. 2: (Colour online) The automaton feels the red potential when $\dot r < 0$, the blue one when $\dot r > 0$. At each inversion point there is a switch with energy loss, until it reaches a point between the two minima where it stops.

**The one-automaton problem.** – The nature of the problem can be better understood studying a single automaton moving in a confining harmonic potential and in a repulsive Coulombian potential that it feels only when the centre of forces falls in its cone of vision. It can be shown that the problem becomes one-dimensional and the radial motion is described by the Hamiltonian

$$H = \frac{\dot r^2}{2} + \frac{L^2}{2r^2} + \frac{\omega^2 r^2}{2} - \ln(r)\vartheta(C), \tag{11}$$

where the cone condition is given by $C = -r\dot r - L\cot\alpha$. When $\alpha = \frac{\pi}{2}$ the cone condition simplifies to $C = -r\dot r$ which means that the switch between the two integrals of motion occurs when an inversion point is reached (the automaton feels a repulsive potential only when approaching the origin).

It is easy to show (fig. 2) that the automaton reaches a relative equilibrium whenever its inversion point is between the minima of the different effective potentials acting when the cone condition is verified or when it is not. To this equilibrium point corresponds a circular orbit for the radial motion. We have verified, using a second-order symplectic integrator, that for any value of $\alpha < \pi$ the asymptotic orbits are closed curves, with an energy lower than the initial one (see fig. 3). An analogous behaviour has been found, always using numerical integration, for the radial motion of the two-automata problem.

**The $N$-automata problem.** – Our numerical study shows that the loss of energy grows with the number $N$ of automata, leading quickly to a "frozen" state with temperature $T = 0$ (we define temperature as the average kinetic energy of the automata). The simulations show that the $N = 2$ problem is the only one with a finite equilibrium temperature (fig. 4).

The most natural way to avoid freezing would have been to introduce some kind of "internal degree of freedom" (and eventually even an internal energy), representing

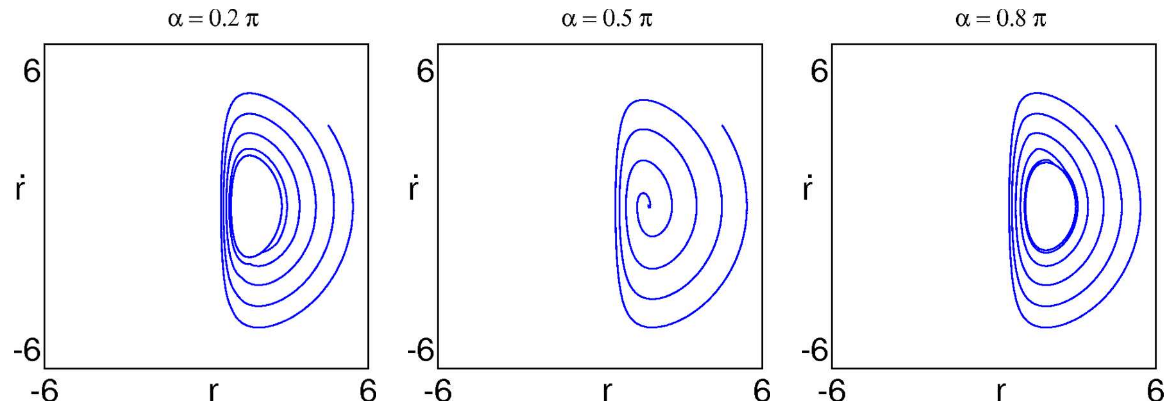


Fig. 3: Phase space trajectories for the radial motion of automata with different values of $\alpha$.

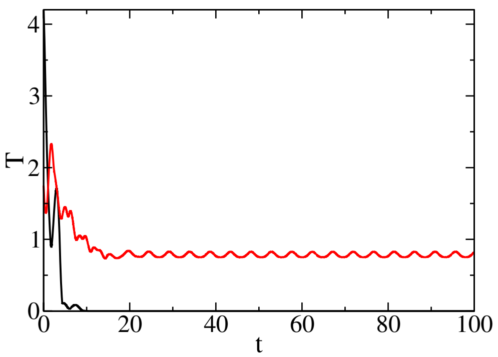


Fig. 4: (Colour online) Time evolution of the overall kinetic energy for a $N=2$ (red line) and $N=3$ (black line) system.

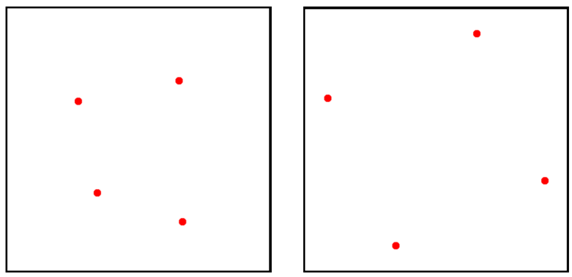

Fig. 5: Equilibrium configurations for $N=4$, $\alpha=\pi/2$, $r_v=\infty$. Left: disordered equilibrium configuration for automata without memory, $\tau=0$. Right: spatially organised configuration obtained using $\tau=0.01$, in which all the automata have (roughly) the same distance from their first neighbours.

the intention of the automaton to reach a given goal. Nevertheless, given our interest in studying the effects of non-Newtonian perception on automaton-to-automaton interaction, and to compare the results to a known physical system (Coulomb oscillators [5]), we decided to represent the tendency of automata to reach the their goal (the centre) only through the confining potential, and to avoid freezing by introducing a *memory* mechanism.

The idea at the base of memory is that an automaton can retain some information about the position of another automaton even after that the latter exits its cone of vision. The most natural assumption would be that the "observer" calculates an approximate trajectory (for example, constant velocity motion) for the observed automaton when it exits the cone of vision. To make the model computationally less expansive we assumed that the observer can actually know the exact position of the observed automaton for a "memory time" $\tau$ (this choice is almost equivalent to calculating an approximate trajectory for low values of $\tau$).

We thus let automaton $A$ feel a repulsion force from $B$ for a time interval $\tau$ after it escapes from its visual cone (if $r<r_v$, where $r$ is the relative distance between automata). By letting $\tau\to\infty$, we recover (after a transient

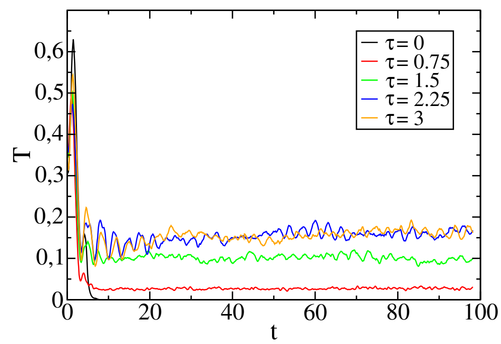


Fig. 6: (Colour online) Time evolution of the "temperature" $T$ of a system with $N=100$ automata, $r_v=1$, $\alpha=0.1$, for different values of $\tau$. Dissipation decreases as $\tau$ grows and for high enough values of $\tau$ the system reaches an equilibrium state after a transient.

phase) the case of $N$ interacting charges without any visual cone, where the total energy and the average kinetic energy are preserved.

We have verified that for very low values of $\tau$ the system goes to an ordered state with $T\approx 0$ (while the spatial distribution was disordered for $\tau=0$, fig. 5), and that

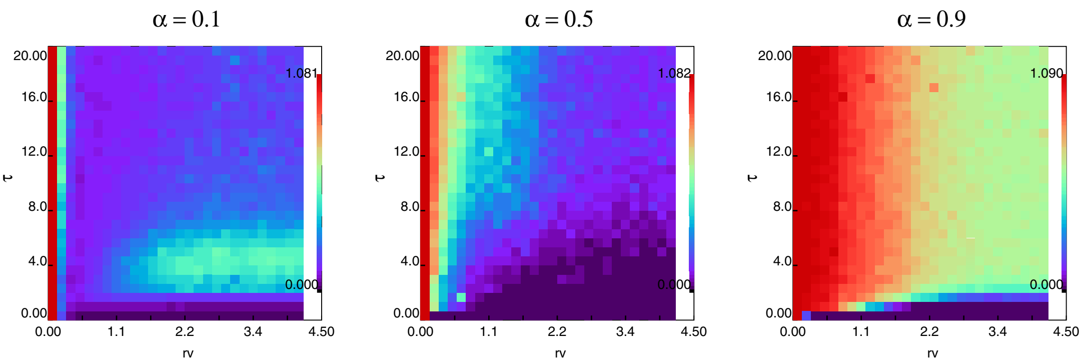


Fig. 7: (Colour online) State diagrams for the gas of automata. The equilibrium values of $T$ are shown for $\alpha = 0.1\pi$ (left), $\alpha = 0.5\pi$ (centre) and $\alpha = 0.9\pi$ (right). $\tau$ is on the $y$-axis, and $r_v$ on the $x$-axis. Red corresponds to high temperature, violet to a low one, as reported on the colour bar. Notice the "high-temperature island" for $\alpha = 0.1\pi$.

for higher values of $\tau$ the system can reach a non trivial equilibrium system, in which the temperature is positive but different from that obtained in the conservative case (fig. 6).

We have performed a throughout numerical investigation of the time evolution for a system of $N = 100$ automata varying the control parameters $\alpha$, $r_v$ and $\tau$. The initial conditions corresponded to to a self-consistent charge distribution for Coulomb oscillators [5], a distribution with radius $\bar{R} = 1.84$ in which the period of oscillation for particles was $\bar{t} \approx 10$. On the basis of these values, to obtain a description of all the features of the system, we studied the following ranges of parameters: $0 \leqslant \alpha \leqslant \pi$, $0 < r_v < 4$ and $0 < \tau < 20$.

This dependence on the parameters (the "state equation" of the gas of automata) is shown in fig. 7. The equilibrium temperature usually decreases when $r_v$ increases at $\alpha$ and $\tau$ fixed, whereas it grows with $\tau$ when the other parameters are fixed. The first rule can be explained considering that, as we stressed before, the introduction of the non-Newtonian effect due to vision always leads to a dissipation of kinetic energy, since the negative power that the automaton feels when approaching another automaton (the force is directed opposite to the velocity) is not completely balanced by the positive power felt when going away. It is clear that the higher is the value of $r_v$, the longer the automata interact and thus dissipate. The second rule is quite obvious since we introduced memory in order to attenuate dissipation.

Regarding the $\alpha$-dependence of temperature we can say that in general, keeping the other parameters fixed, the temperature is maximum in the $\alpha = 0$ and $\alpha = \pi$ cases (*i.e.*, in conservative systems), while it attains a minimum for $\alpha = \pi/2$.

Exceptions to these rules are found for low values of $\alpha$, where we found "islands" of high temperature for certain ranges of $\tau$. This happens when memory allows the automaton to feel (as can be explained on the base of geometric considerations), in certain situations, the repulsive force mainly directed along its velocity

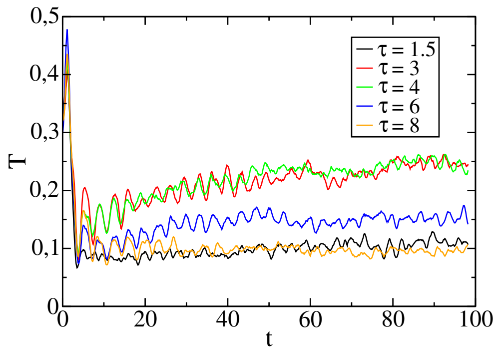


Fig. 8: (Colour online) Time evolution of the "temperature" $T$ of a system with $N = 100$ automata, $r_v = 2$, $\alpha = 0.1$, for different values of $\tau$. Inside a given $\tau$ range the system reaches a higher equilibrium value.

(positive power) and thus attenuates the damping process. An analysis of the transition to equilibrium in the system for these particular values of the parameters shows the presence not only of a dissipative transient (as for all the other values) but also of a second transient during which a part of the lost energy is regained (fig. 8).

Since our system resembles (and is equal to in the $\alpha = \pi$ and $r_v = \infty$ case) a system of Coulomb oscillators, which is known to assume uniform crystal configurations when frozen to $T = 0$ [6], and since our system naturally freezes during time evolution, we studied also the emergence of these organised structures and verified that they are actually present for low values of $\tau > 0$ and for large enough values of $r_v$ (both memory and a large enough range of vision are necessary for the system to organise, but when $\tau$ is too large the system does not freeze). In particular, since we had noticed that in these structures the automata where roughly located at uniform distances from their first neighbours, we defined a "disorder parameter"

$$\gamma = \frac{\Delta d_f}{\langle d_f \rangle} \tag{12}$$

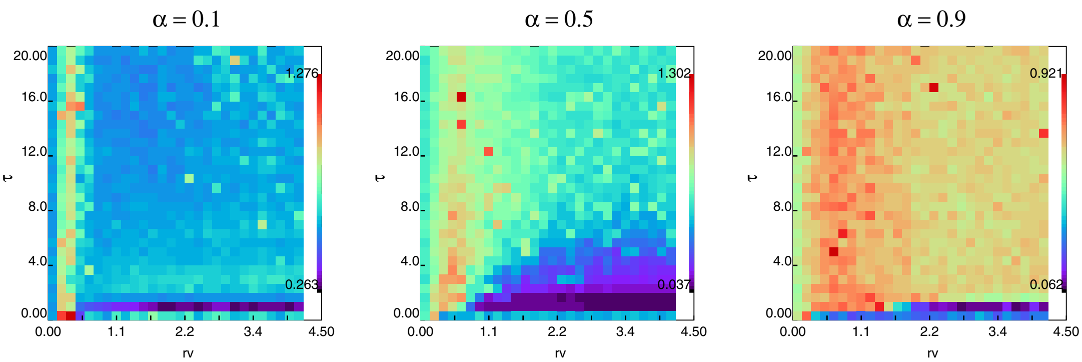


Fig. 9: (Colour online) Equilibrium values of $\gamma$ are shown for $\alpha=0.1\pi$ (left), $\alpha=0.5\pi$ (centre) and $\alpha=0.9\pi$ (right). $\tau$ is on the $y$-axis, and $r_v$ on the $x$-axis. Red corresponds to a disordered structure and violet to an ordered one, as reported on the colour bar. Ordered crystals are present for low values of $\tau>0$ and for large enough values of $r_v$. The structure is almost perfect if $\alpha=0.5\pi$, while it is very poor for low $\alpha$.

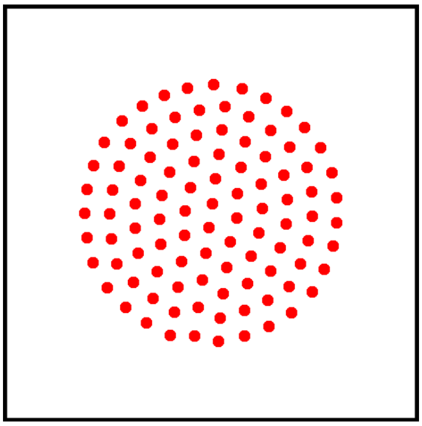

Fig. 10: Equilibrium configuration for $N=100$, $r_v=\infty$, $\alpha=\pi/2$ and $\tau=0.1$.

(where $d_f$ is the distance to the first neighbour and $\Delta d_f$ its mean squared deviation) that goes to zero in case of a perfect uniform crystal. The equilibrium values of $\gamma$ are shown in fig. 9, while one of these crystals is shown in fig. 10.

**Conclusions.** – The automata model we presented is intended as a very naive model of people moving in an open space with a single attracting point (the centre of the harmonic potential), which renders the system symmetric by rotation. Obviously both the pure misanthropic behaviour of the automata and the harmonic attraction to the centre are not realistic enough to describe human or animal behaviour, but they allow us to make a comparison with a known conservative model (Coulomb oscillators, see [5]), and focus on the non-Newtonian effects due to sensory perception (vision).

Another possible application of our model is the study of an "avenue", in which we consider two groups of automata moving in opposite directions and study the minimal request for the emergence of organised patterns [7].

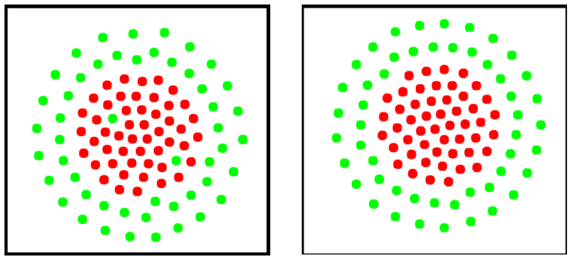

Fig. 11: (Colour online) Metastable (left) and stable (right) $T\approx 0$ configurations for automata split in two groups with different "social radii" $r_s$. Green automata have $r_s=2$, red ones $r_s=1.5$.

Our preliminary study shows that in the presence of a constant force field and a dissipation the two groups reach a constant opposite velocity, in the absence of mutual interactions. The repulsive mutual interaction causes self-organisation phenomena with the formation of streams, whose properties depend on the vision parameters.

This work might also be used as the basis for a more complex model in which we can introduce more realistic social forces that are attractive over a given social radius [4], and also some heterogeneity by sampling the vision parameters and the mutual force strength, in order to simulate the approach to the equilibrium of a system of individuals with contrasting goals (fig. 11 shows a metastable and a stable equilibrium configuration for such a system).

* * *

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